

# Hypergames and full completeness for system $F$

## (ROUGH DRAFT)

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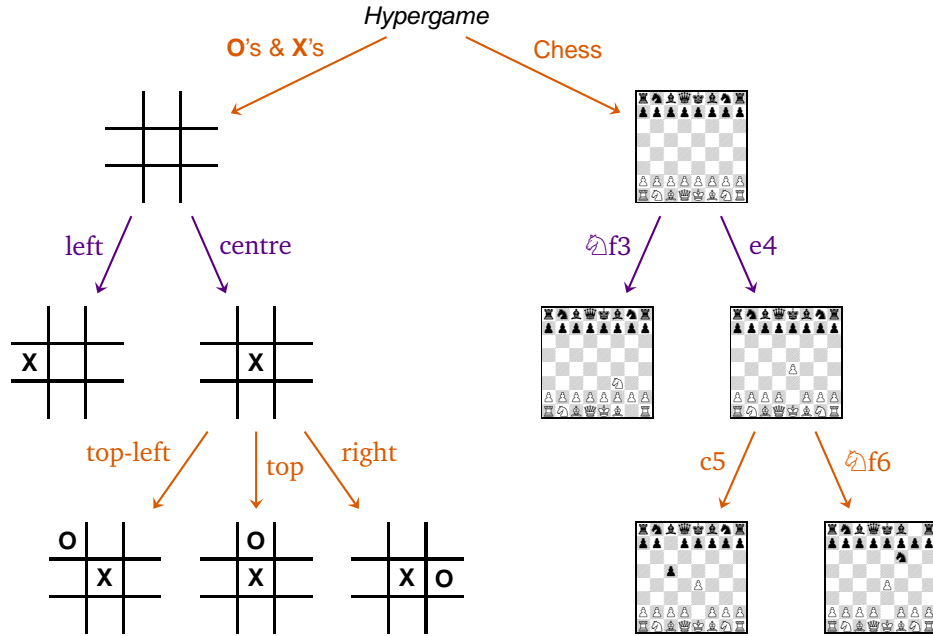
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This paper reviews the fully complete *hypergames* model of system  $F$ , presented a decade ago in the author's thesis. Instantiating type variables is modelled by allowing “games as moves”. The uniformity of a quantified type variable  $\forall X$  is modelled by *copycat expansion*:  $X$  represents an unknown game, a kind of black box, so all the player can do is copy moves between a positive occurrence and a negative occurrence of  $X$ .

This presentation is based on slides for a talk entitled “Hypergame semantics: ten years later” given at *Games for Logic and Programming Languages*, Seattle, August 2006.

## 1 Introduction

Zwicker's *Hypergame* [Zwi87] is an alternating two-player game: one player chooses any alternating game  $G$  which terminates<sup>1</sup> (e.g. “O's & X's” or Chess<sup>2</sup>), then play proceeds in  $G$ .<sup>3</sup>



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<sup>1</sup>Every legal sequence of moves is finite.

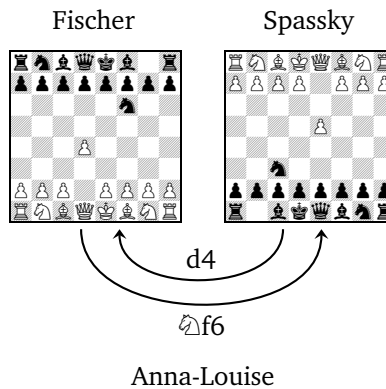
<sup>2</sup>To ensure termination, assume a draw is forced upon a threefold repetition of a position (a variant of a standard rule).

<sup>3</sup>The question “Does *Hypergame* terminate?”, the *Hypergame paradox*, amounts to a hereditary form of Russell's paradox, known as Mirimanoff's paradox [Mir17]: “Is the set of well-founded sets well-founded?”. (Each ‘paradox’ is illusory, being merely due to the lack of formal definition of “game” or “set”.)

At the *Imperial College Games Workshop* in 1996, the author illustrated how hypergames — games in which games can be played as moves — can model languages with universal quantification. Originally implemented in [Hug97] for Girard’s system F [Gir71, GLT89], the idea is quite general, and has been successfully applied to affine linear logic [MO01, Mur01] and Curry-style type isomorphisms [dL06].

### 1.1 Universally quantified games

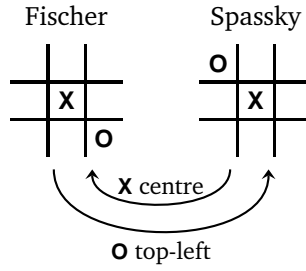
Recall the little girl Anna-Louise who wins one point out of two in a “simultaneous display” against chess world champions Spassky and Fischer [Con76, Theorem 51]. She faces Spassky as Black and Fischer as White, and copies moves back and forth, indirectly playing one champion against the other. When Spassky opens with the Queen’s pawn d4, she opens d4 against Fischer; when Fischer responds with the King’s knight  $\text{♞f6}$ , she responds  $\text{♞f6}$  against Spassky, and so on.



We shall write  $G \rightarrow G$  for such a simultaneous display with a game  $G$  (so Anna-Louise played the game  $\text{Chess} \rightarrow \text{Chess}$  above, as second player, against the Fischer-Spassky team).<sup>4</sup>

Observing that her copycat strategy is not specific to chess, Anna-Louise declares that she will tackle the Fischer-Spassky team in a more grandiose spectacle: she will give them an additional first move, to decide the game for simultaneous display. For example, the Fischer-Spassky team might choose Chess, thereby opting for the simultaneous display  $\text{Chess} \rightarrow \text{Chess}$ , and play continues as above. Or they might choose **O's & X's**, opting for the simultaneous display **O's & X's**  $\rightarrow$  **O's & X's**, and open with **X** in the centre of Spassky’s grid; Anna-Louise copies that **X** accross as her opening move on Fischer’s grid; Fischer responds with **O** in (his) top-left; Anna copies this **O** back to Spassky; and so on:

<sup>4</sup>Conway writes  $-G + G$ , or  $G - G$  [Con76, Chapter 7]. Later on, we shall add a form of backtracking to our games so that Anna-Louise may restart the game with Fischer as many times as she likes, corresponding to the intuitionism of the arrow  $\rightarrow$  of system F, in which a function may read its argument any number of times [Lor60, Fel85, Coq91, HO00]. To maintain the focus on universal quantification, here in the introduction we shall ignore the availability backtracking.

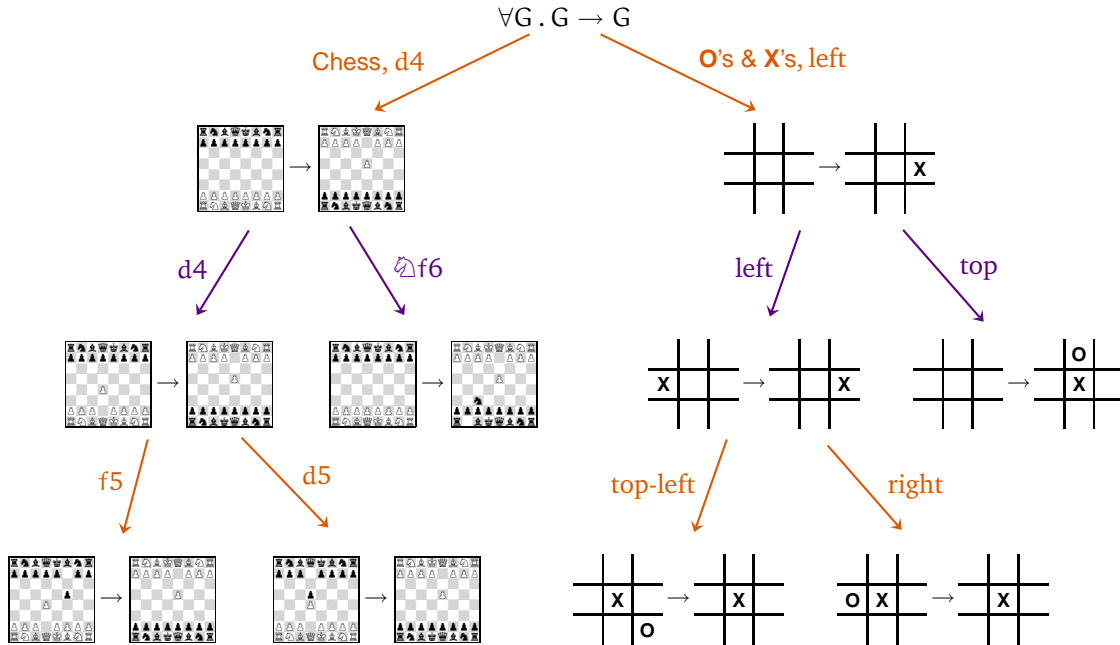


Anna-Louise

The key novelty of [Hug97] was to define this as a formal game, a *hypergame* or *universally quantified game*, which we shall write as

$$\forall G . G \rightarrow G$$

The tree of  $\forall G . G \rightarrow G$  is illustrated below. Similar in spirit to Zwicker's hypergame<sup>5</sup>, it differs in the fact that the first player not only chooses  $G$  but also plays an opening move  $m$  in  $G$ . We call such a compound move (importing a game, and playing a move in a game) a *hypermoves*.



## 1.2 Self-reference (without paradox)

In the tree above, we have shown two cases for instantiating  $G$  in the hypergame  $H = \forall G . G \rightarrow G$ , either to Chess or to  $O$ 's &  $X$ 's. But it is also possible to instantiate  $G$  to a hypergame, or indeed, to  $H$  itself. We consider this case below. The initial state is:

<sup>5</sup>The author was unaware of Zwicker's work while preparing [Hug97], hence the lack of reference to Zwicker in that paper, and in the author's thesis [Hug00].

Fischer                  Spassky

$\forall G . G \rightarrow G$

Anna-Louise

Fischer and Spassky begin by importing a game for  $G$ , in this case,  $H = \forall G . G \rightarrow G$  itself, yielding a simultaneous display of  $H$ :

Fischer                  Spassky

$H \rightarrow H$

Anna-Louise

In other words, we have:

Fischer

Spassky

$\forall G_1 . G_1 \rightarrow G_1 \quad \rightarrow \quad \forall G_2 . G_2 \rightarrow G_2$

Anna-Louise

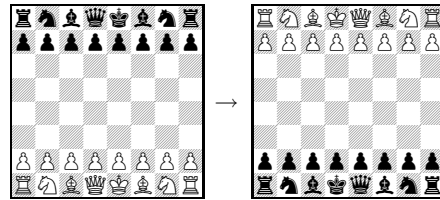
The local bound variable  $G$  is renamed in each component to clarify the evolution of the game below.<sup>6</sup> As in the simultaneous display  $\text{Chess} \rightarrow \text{Chess}$ , where Spassky opened with a move on his chessboard, here in  $H \rightarrow H$  Spassky must complete the opening hypermove by playing a move on his copy of  $H$ . Since  $H = \forall G_2 . G_2 \rightarrow G_2$  is a hypergame, opening  $H$  requires importing *another* game, instantiating  $G_2$ . Suppose he chooses Chess for  $G_2$ :

Fischer

Spassky

$\forall G_1 . G_1 \rightarrow G_1$

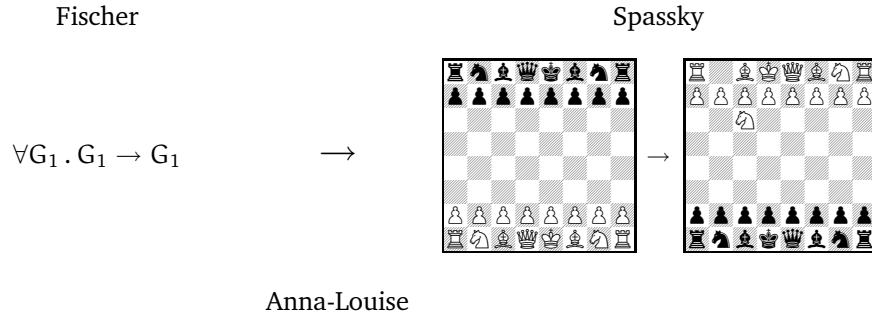
$\rightarrow$



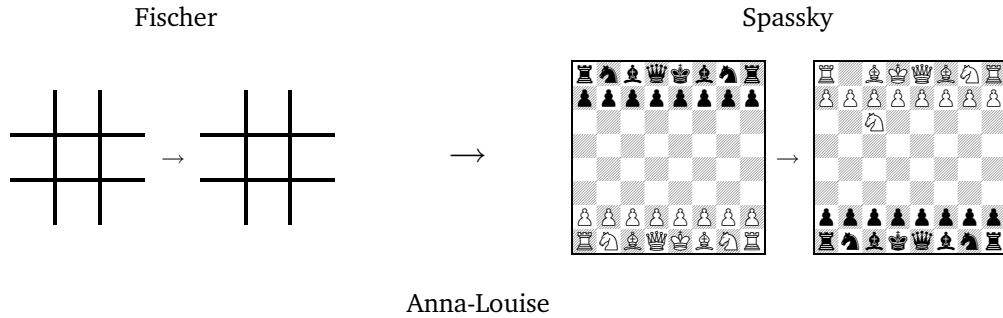
Anna-Louise

<sup>6</sup>The scope of  $\forall G_1$  in the diagram does not extend past the central arrow  $\rightarrow$ . In other words, formally the game played by Anna-Louise is  $(\forall G_1 . G_1 \rightarrow G_1) \rightarrow (\forall G_2 . G_2 \rightarrow G_2)$ .

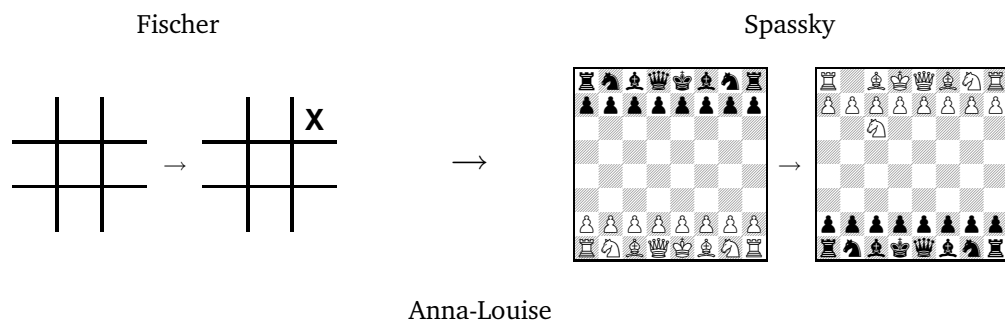
Now Spassky has his own local simultaneous display  $\text{Chess} \rightarrow \text{Chess}$ . To complete his opening (hyper)move on the overall game, he must open this chess display. Suppose he plays  $\text{♟f3}$  (necessarily on the right board, where it is his turn since he has White):



Now it is Anna-Louise's turn. She has three options: (1) respond to Spassky as Black on the rightmost chess board, (2) respond to Spassky as White on the other chess board, or (3) play an opening move against Fischer in  $\forall G_1 . G_1 \rightarrow G_1$ . We consider the last case, since it is the most interesting. Suppose Anna-Louise chooses to import **O's & X's** for  $G_1$ :

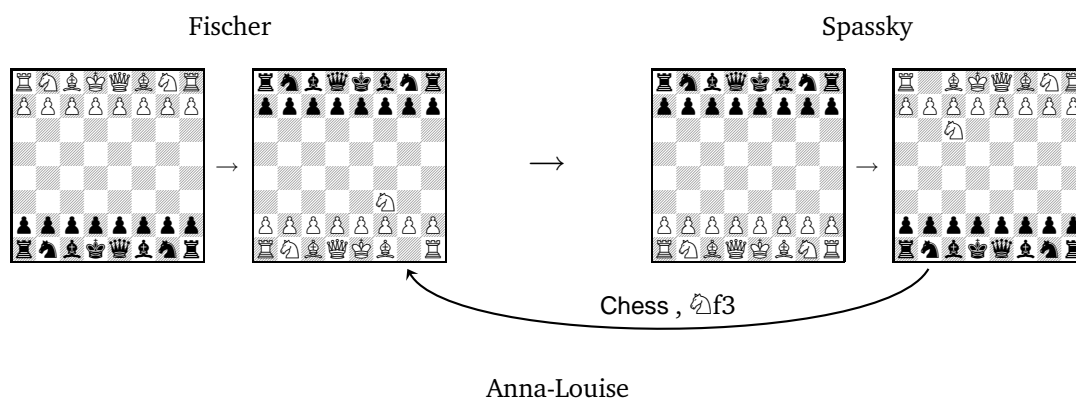


Now Fischer has his own local simultaneous display  $\text{O's \& X's} \rightarrow \text{O's \& X's}$ . For Anna-Louise to complete her hypermove, she must play a move on  $\text{O's \& X's} \rightarrow \text{O's \& X's}$  (necessarily in the right of the two grids, the one in which it her turn). Suppose she plays her **X** top-right:

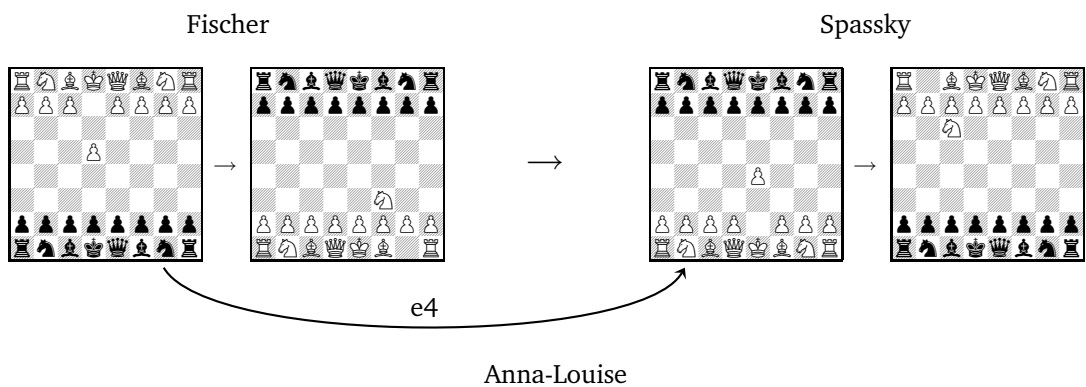


Fischer responds either with an **O** in the same grid, or with an **X** in the empty grid, and play continues in the two local simultaneous displays, **O**'s & **X**'s  $\rightarrow$  **O**'s & **X**'s against Fischer and Chess  $\rightarrow$  Chess against Spassky.

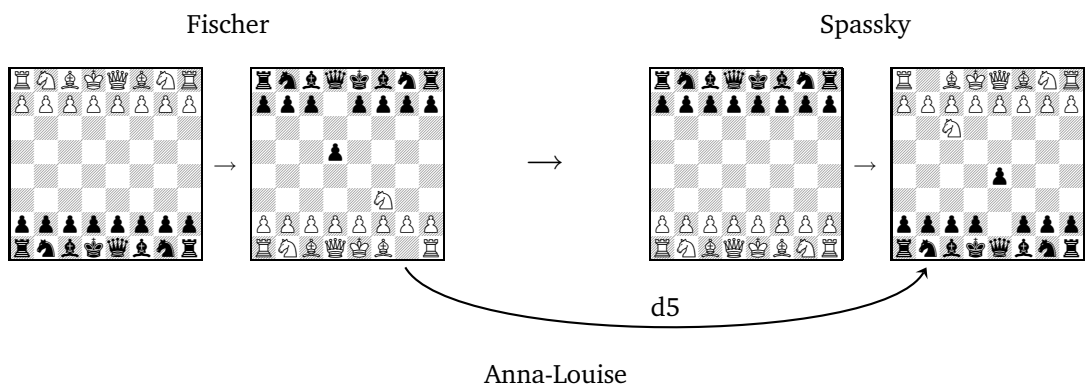
But to remain consistent with her copycat strategy, Anna-Louise must mimic Spassky. Instead of importing **O**'s & **X**'s for  $G_1$  against Fischer, she must import Chess and open with the White move  $\text{Nf3}$ , exactly as Spassky did:



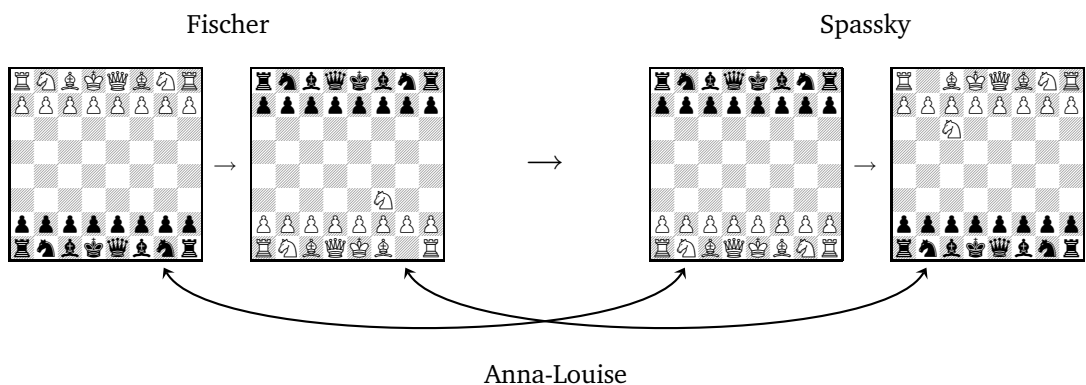
Fischer might now open his other board with e4, which Anna-Louise would copy back to the corresponding board against Spassky:



Or perhaps Fischer responds with Black in the rightmost of his pair of boards, with d5, which Anna-Louise copies to Spassky:



Either way, she continues to copy moves between the four boards according to the following geometry of copycat links:



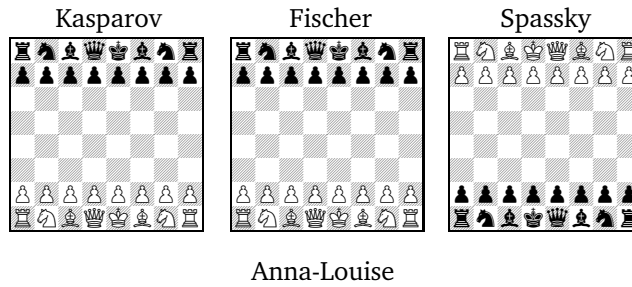
This copycat strategy corresponds to the polymorphic identity system F term

$$\Lambda G. \lambda g^G. g$$

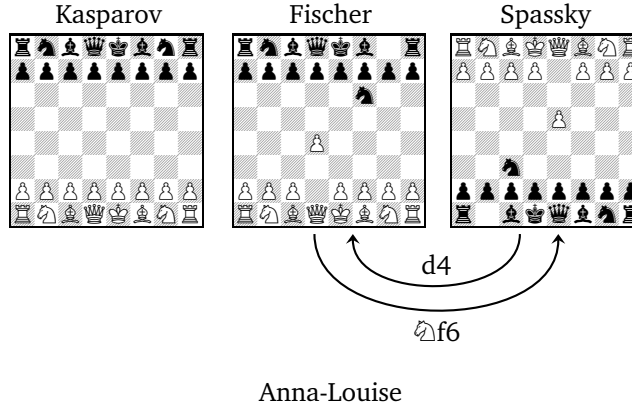
of type  $\forall G. G \rightarrow G$ .

### 1.3 Uniformity

Consider again the original Fischer-Spassky simultaneous display, with chess. Add Kasparov to the team, playing Black.

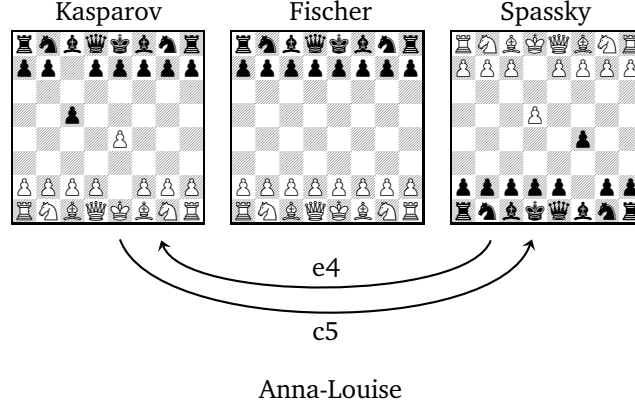


Anna-Louise has two distinct ways to guarantee picking up a point. Either she copies moves between Spassky and Fischer, as before, while ignoring Kasparov (never playing a move against him),



or she copies moves between Spassky and Kasparov, ignoring Fischer:





We shall write this triple simultaneous display as  $\text{Chess} \rightarrow \text{Chess} \rightarrow \text{Chess}$ , and more generally, for any game  $G$ , as  $G \rightarrow G \rightarrow G$ .<sup>7</sup>

Now consider the universally quantified form of this game, the hypergame

$$\forall G . G \rightarrow G \rightarrow G.$$

As with  $\forall G . G \rightarrow G$  discussed above, the Kasparov-Fischer-Spassky team, KFS, now has the right to choose the game of the triple simultaneous display, as part of their opening (hyper)move. We shall say that Anna-Louise's strategy is **uniform** in this setting if

- irrespective of the game chosen by KFS, she always ignores the same player, Kasparov or Fischer.

Otherwise her strategy is **ad hoc**. For example, her strategy would be ad hoc if, when KFS chooses Chess, she ignores Kasparov and copies chess moves between Fischer and Spassky, but when KFS chooses **O's & X's**, she ignores Fischer and copies **X** and **O** moves between Kasparov and Spassky. In this case the geometry of her move copying depends on the game imported by FKS: she is not treating  $G$  as a “black box”.

There are only two uniform strategies for Anna-Louise: either she always copies between *Kasparov* and *Spassky*, ignoring *Fischer*, or she always copies between *Fischer* and *Spassky*, ignoring *Kasparov*. These correspond to the system F terms

$$\begin{aligned} \Lambda G . \lambda k^G . \lambda f^G . k \\ \Lambda G . \lambda k^G . \lambda f^G . f \end{aligned}$$

respectively, of type

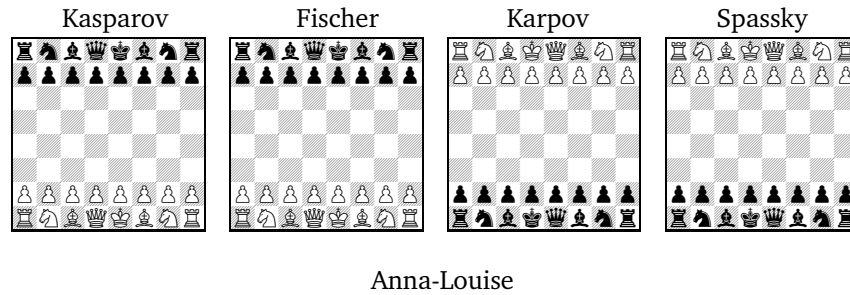
$$\forall G . G \rightarrow G \rightarrow G,$$

where the variable  $k$  corresponds to Kasparov and  $f$  corresponds to Fischer.

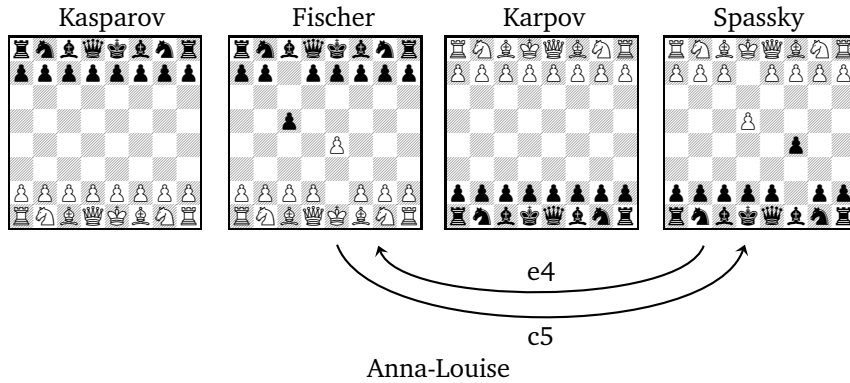
More generally, with multiple bound  $\forall$  variables and more complicated game imports, we shall take uniformity to mean that the links Anna-Louise sets up between components (such as the  $\text{Kasparov} \leftrightarrow \text{Spassky}$  or  $\text{Fischer} \leftrightarrow \text{Spassky}$  links above) must be independent of the games imported by the opposing team: these imported games are impenetrable “black boxes”.

<sup>7</sup>Again with the backtracking caveat: see footnote 4.

**Fixed links.** Uniformity as independence from the particular games imported by the opposing team will include independence from the not only the *identity* of those games, but also from their *state*. This will ensure that the geometry of Anna-Louise’s copycat play remains constant over time: once she has committed to linking one component to another, she must stick with that link for the rest of the hypergame. To illustrate this aspect of uniformity, consider the quadruple chess simultaneous display with Kasparov and Fischer playing Black, and Karpov and Spassky playing White:

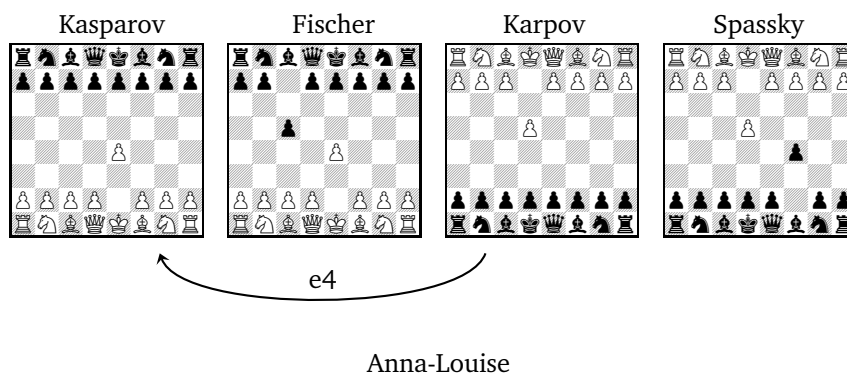


We shall write  $\text{Chess} \times \text{Chess} \rightarrow \text{Chess} \times \text{Chess}$  for this simultaneous display.<sup>8</sup> Suppose Spassky begins with e4. Anna-Louise, playing copycat, has a choice between copying this move to Fischer or to Kasparov. Suppose she copies it to Fischer, who responds with c5, which she duly copies back to Spassky:

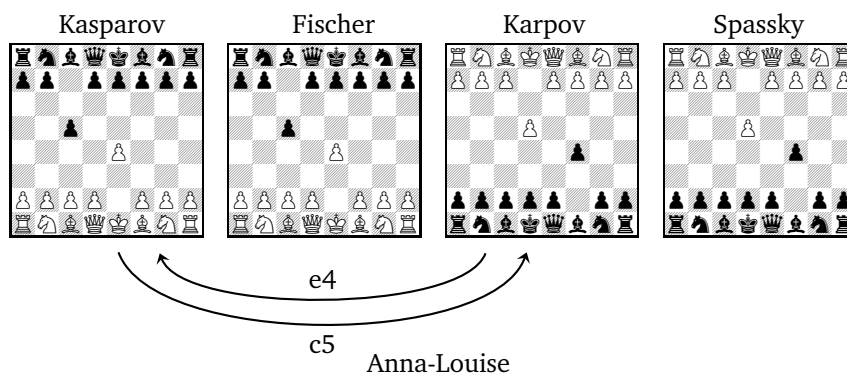


Suppose Karpov opens his game with the very same move as Spassky, e4, which Anna-Louise copies across to Kasparov (the only destination where this move makes sense):

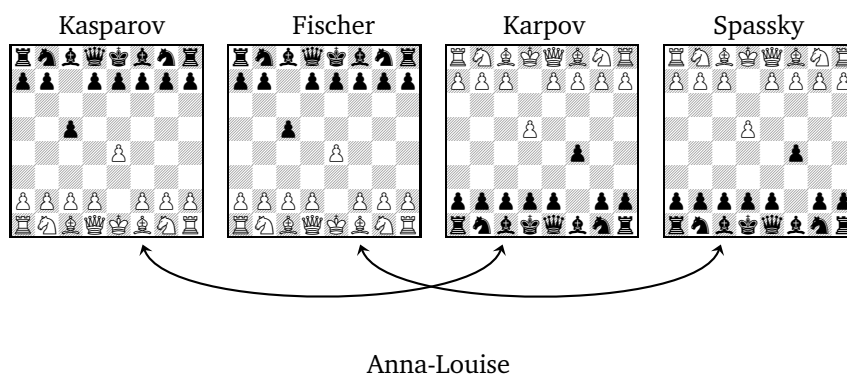
<sup>8</sup>With the backtracking caveat: see footnote 4.



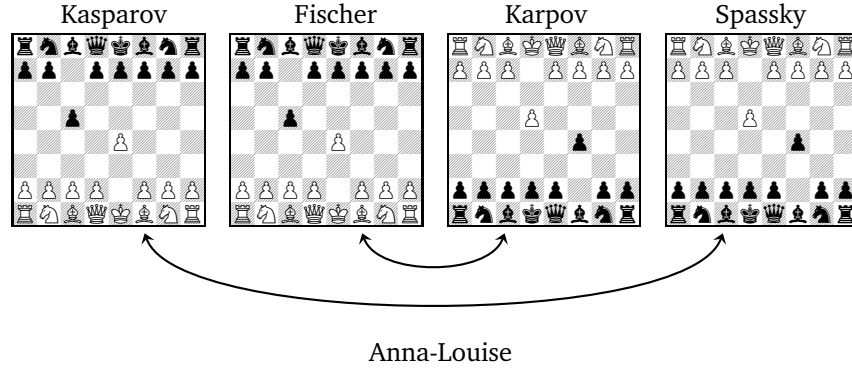
Kasparov responds with the same move as Fischer, c5, which Anna-Louise copies back to Karpov:



So far, Anna-Louise has linked Spassky with Fischer, and Karpov with Kasparov:



By (contrived) coincidence, both pairs of linked boards happen to have reached exactly the same state. Therefore from this point onwards, Anna-Louise could change the linkage, linking Kasparov with Spassky, and Karpov with Fischer:



For example, should Karpov respond with  $\text{Nf3}$ , she would copy that move across to Fischer, then continue copying between Fischer and Karpov, and between Kasparov and Spassky.

She could do this “relinking” for any game  $G$ , not just Chess, on  $G \times G \rightarrow G \times G$ : no matter what the game  $G$  is, she could link the first and third  $G$ , and link the second and fourth  $G$ , but if a point is reached in which all four copies of  $G$  have the same state, she switches the linkage, as in the chess example above. If she consistently does this for all  $G$ , she has a strategy on the hypergame  $\forall G. G \times G \rightarrow G \times G$  which, in some fashion, does not depend on  $G$ . Such “relinking” strategies do not correspond to system  $F$  terms, and are eliminated from the model by our uniformity condition: independence from  $G$  means independence not only from the *identity* of  $G$ , but also from the *state* of  $G$ .

#### 1.4 Negative quantifiers

Linear polymorphism was modelled in [Abr97] using a universal notion of the games in [AJ94, AJM00]. Full completeness failed for types with negative quantifiers. In this subsection we illustrate how the hypergames model successfully treats negative quantifiers.

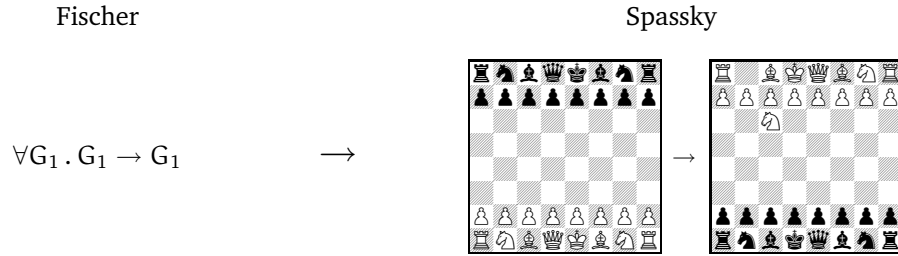
The polarity of a quantifier in a type is positive or negative according to the number of times it is to the left of an arrow (in the syntactic parse tree of the type): positive if even, negative if odd. For example,  $\forall X$  is positive in  $\forall X.T$  and  $\forall Y.\forall X.T$ , negative in  $(\forall X.U) \rightarrow T$  and  $(V \rightarrow \forall X.U) \rightarrow T$ , and positive in  $((\forall X.U) \rightarrow V) \rightarrow T$ .

Consider the simultaneous display  $H \rightarrow H$  where  $H$  is the hypergame  $\forall G. G \rightarrow G$ :

$$\begin{array}{ccc}
 \text{Fischer} & & \text{Spassky} \\
 \forall G_1 . G_1 \rightarrow G_1 & \longrightarrow & \forall G_2 . G_2 \rightarrow G_2
 \end{array}$$

Anna-Louise

Fischer’s quantifier  $\forall G_1$  is negative.<sup>9</sup> To kick off, Spassky must open the game  $\forall G_2 . G_2 \rightarrow G_2$  in front of him. This is a hypergame, universally quantified, so he must begin by instantiating  $G_2$ . He chooses  $G_2 = \text{Chess}$ , and opens  $\text{♟f3}$  on the board where he has White:



Anna-Louise

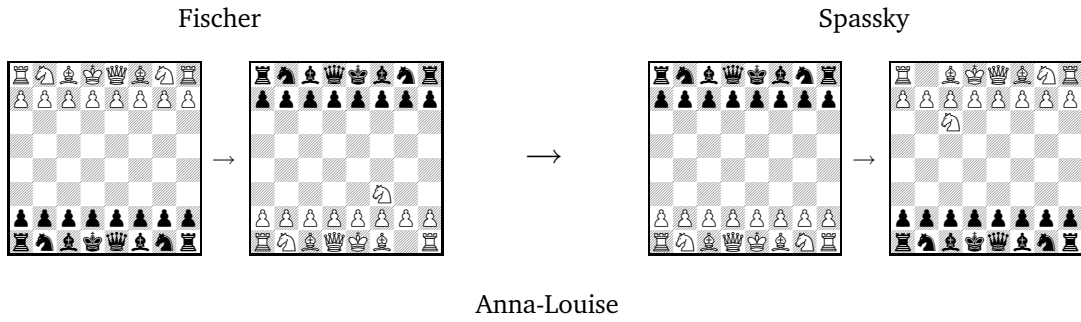
We shall consider three of the copycat strategies available to Anna-Louise from this point:

Strategy	Anna Louise...	Corresponding term of type $H \rightarrow H$
$\iota$	... copies what Spassky did across to Fischer: import Chess and play $\text{♟f3}$	$\lambda h^H . h$
$\sigma$	... plays copycat in Spassky’s local chess display, “playing Spassky against himself”	$\lambda h^H . \Lambda G . \lambda g^G . g$
$\tau$	... imports $G_1 = \text{Chess} \rightarrow \text{Chess}$ against Fischer, then copies moves between the six resulting boards, along three “copycat links”	$\lambda h^H . \Lambda G . \lambda g^G . h_{G \rightarrow G} (\lambda x^G . x) g$

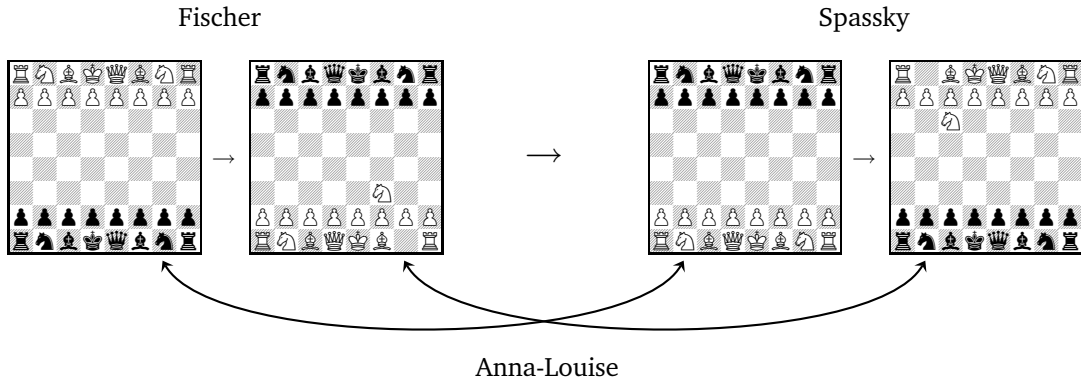
The notation  $h_U$  in the third term denotes the application of  $h$  to the type  $U$ .

**The first copycat strategy  $\iota$ .** Anna-Louise opens the hypergame  $\forall G_1 . G_1 \rightarrow G_1$  in front of Fischer by mimicking Spassky: she imports Chess for  $G_1$  and opens with  $\text{♟f3}$  as White:

<sup>9</sup>The scope of  $\forall G_1$  in the diagram does not extend past the central arrow  $\longrightarrow$ .



She then copies moves between the four boards according to the following geometry of copycat links:

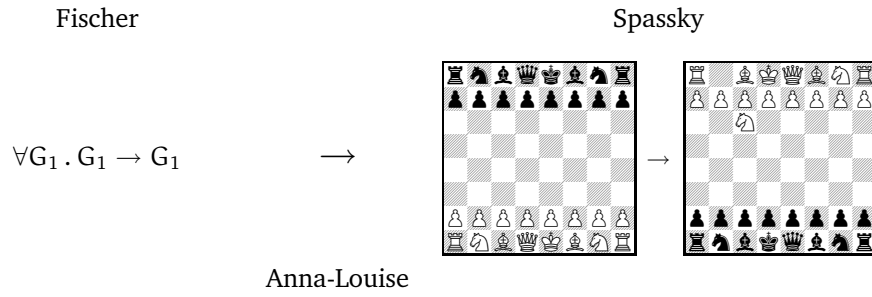


This copycat strategy  $\iota$  corresponds to the identity system F term

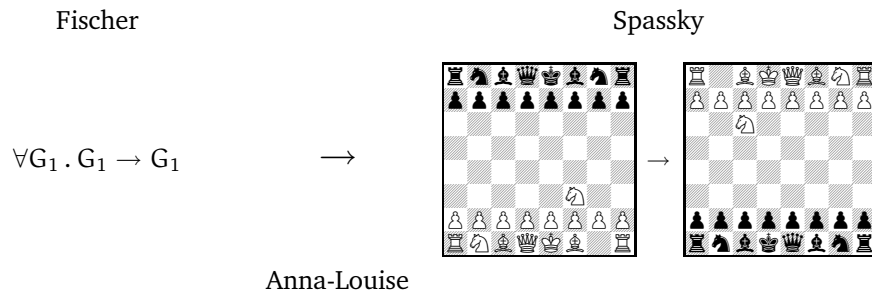
$$\lambda h^H. h$$

of type  $H \rightarrow H$ . (Recall  $H = \forall G. G \rightarrow G$ .) The same strategy models the  $\eta$ -expanded variant  $\lambda h^H. \lambda g^G. \lambda g_G. h_G g$ .

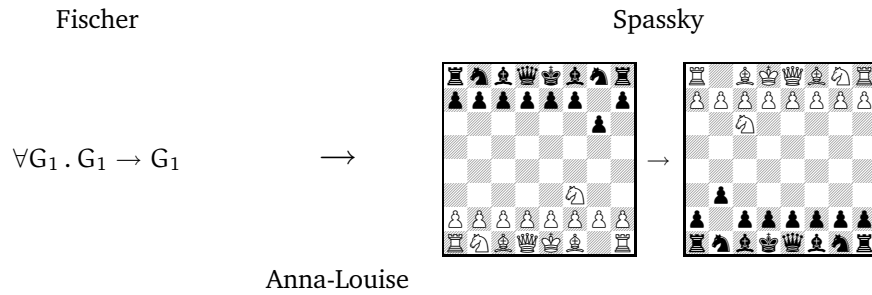
**The second copycat strategy  $\sigma$ .** The second copycat strategy  $\sigma$  “plays Spassky against himself”. Recall the state after Spassky’s opening move:



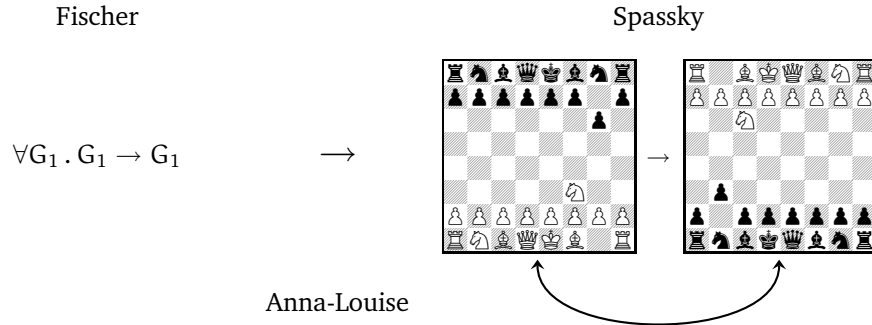
Spassky has just imported Chess and opened with the White move  $\text{Nf3}$ . In this scenario Anna-Louise copies that move locally, to the other board in front of Spassky:



Spassky may respond with g6 as Black, which Anna-Louise copies back to the other board:



She continues to copy moves along the following copycat link, leaving Fischer to forever twiddle his thumbs:

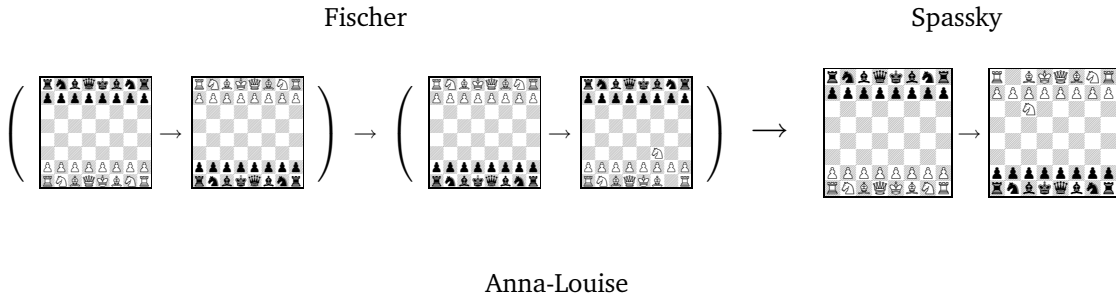


This copycat strategy corresponds to the system F term

$$\lambda h^H . \Lambda G . \lambda g^G . g$$

of type  $H \rightarrow H$ . (Recall  $H = \forall G . G \rightarrow G$ .) Fischer's eternal thumb twiddling corresponds to  $h$  not showing up in the body of the term.

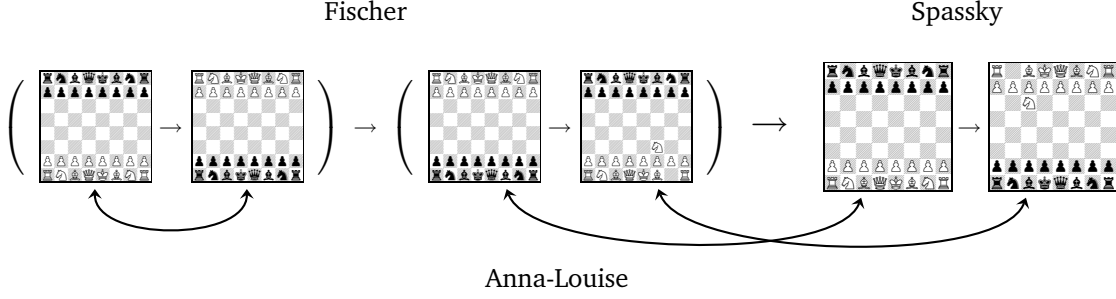
**The third copycat strategy  $\tau$ .** The third copycat strategy  $\tau$ , like the first, the identity  $\iota$ , responds to Fischer. However, instead of importing Chess for  $G_1$  against Fischer, as in  $\iota$ , Anna-Louise imports a simultaneous chess display  $\text{Chess} \rightarrow \text{Chess}$  for  $G_1$ .<sup>10</sup>



As shown above, Anna-Louise copies Spassky's  $\text{♟f3}$  onto the fourth board against Fischer. She continues with the following geometry of copycat links:

<sup>10</sup>As usual, the large arrow  $\longrightarrow$  between Fischer and Spassky binds most strongly (so we can omit brackets around the left four boards).





On the right four boards she continues just as on the four boards of the identity  $\iota$ . If Fischer responds as Black on the fourth board, she copies this to the last board against Spassky, and if Fischer opens as White on the third board, she copies this to open the other board against Spassky.

On the left two boards she “plays Fischer against himself”. If Fischer opens with White on the second board, she copies this to him on the first board; if Fischer responds as Black on the first board, she copies that back to the second board. This corresponds to the first argument  $\lambda x^G.x$  of  $h_{G \rightarrow G}$  in the term

$$\lambda h^H. \Lambda G. \lambda g^G. h_{G \rightarrow G} (\lambda x^G. x) g$$

associated with this strategy.

Note that all three of the above copycat strategies are uniform: had the imported game been **O's & X's** instead of Chess, Anna-Louise would have copied the moves around in exactly the same geometry. In the third strategy she would have imported **O's & X's**  $\rightarrow$  **O's & X's** for  $G_1$  against Fischer. This strategy always imports  $K \rightarrow K$  against Fischer, whatever the game  $K$  imported by Spassky. The geometry of Anna-Louise's six copycat links is independent of  $K$ .

## 1.5 Other conceptual ingredients of the model

*This subsection may be somewhat abstruse for readers not already familiar with game semantics; consider skipping to Section 2 below, without loss of continuity.*

So far in this introduction we have sketched the following ingredients of our model:

- Hypergames: games as moves, to model universal quantification/instantiation.
- Self-reference: hypergames can be imported into hypergames, and a hypergame may even be imported into itself.
- Uniformity: the shape of Anna-Louise's play, in terms of how we copy moves around, cannot depend on the choices of games imported by the opposing team: she must treat those games as “black boxes”. Once two (sub)games are linked by copycat, she cannot change that link.

The following additional ingredients come from prior (first-order, unquantified) work:

- Backtracking. We permit moves to be taken back during play, corresponding to the fact that a system  $F$  function can call its argument an arbitrary number of times. Backtracking was used by Lorenzen [Lor60, Fel85] for modelling proofs of intuitionistic logic, by Coquand [Coq91, Coq95], and by Hyland and Ong [HO00].

- **Innocence.** Following Coquand [Coq91, Coq95], Hyland-Ong [HO00] and Nickau [Nic96], strategies depend only on a restricted “view” of the history of play.
- **Interaction.** We use Coquand-style interaction between backtracking strategies to model normalisation of system F terms, specifically, the refinement by Hyland and Ong of this interaction in a lambda calculus (cartesian closed) setting.
- **Liveness.** A strategy must always be able to make a move (coined *liveness* by Conway [Con76]).
- **Copycat condition.** We impose (a restriction of) Lorenzen’s condition [Lor60] for dialogues listed by Felscher as (D10) [Fel85], which requires that an atomic formula (or in the present system F context, a type variable) be “asserted” by Anna-Louise only if, within her view, the opposing team has just asserted it.<sup>11</sup>

These additional ingredients relate to quantifiers:

- **Copycat expansion.** Technically, uniformity will be implemented by *copycat expansion* [Hug06], similar to Felscher’s *skeleton expansion* [Fel85, Fel01] (and equivalent to the condition in [Hug97, Hug00]): whenever a strategy includes a play (accepts a move sequence)  $p$ , with a variable  $X$  imported by the opponent into a quantified variable, then for all types  $T$ , all variants of  $p$  obtained by substituting  $T$  for  $X$  and playing copycat between appropriate instances of  $T$  are also in the strategy.<sup>12</sup>
- **Compactness.** A strategy is determined by a finite “skeleton”, which expresses only the copycat links between components.

The main theorem is that the map from system F terms to strategies (satisfying the above properties) is surjective. A surjectivity theorem of this kind for simply typed  $\lambda$ -calculus is given in [Plo80], but since [AJ92] such a result in a logical setting has often come to be referred to as *full completeness*, when it includes a semantic notion of composition.

### 1.5.1 Modular construction of games

We shall define system F games modularly. First we define a transition system whose states are system F types, and whose transition labels are hypermoves. The hypermoves involve instantiating quantifiers in the states (just as the examples above involved instantiating quantifiers during play).

Every transition system determines a forest (disjoint union of trees): its set of non-empty traces. Every forest can be interpreted as an *arena*, in the sense of Hyland and Ong [HO00].

Following Hyland and Ong, every arena defines a game, with backtracking. The (hyper)game we associate with a system F type will be such a backtracking arena-game. Since we use arena games, interaction of strategies (composition) is precisely the Hyland-Ong interaction.

The underlying first-order composition allows us to relate the composition to an underlying untyped lambda calculus machine, as in [DHR96], upon erasing the system F type information. In other words, the composition, when viewed as acting on  $\eta$ -long  $\beta$ -normal forms (representing innocent view functions), corresponds to (a) erasing the system F type information, (b) computing with the abstract machine [DHR96] on the underlying untyped lambda term, then (c) replacing

<sup>11</sup>I was unaware of Lorenzen’s (D10) at the time I wrote [Hug97, Hug00].

<sup>12</sup>I was unaware of Felscher’s skeleton expansion at the time I wrote [Hug97, Hug00].

type information.<sup>13</sup> If we erase the type information but stay in the model (*i.e.*, we don't look at the lambda terms), then we are just composing strategies in a naive games model of untyped lambda calculus. The underlying transition system of the untyped lambda game has a single state and every integer  $i \geq 1$  as transition labels. These integer labels are precisely the result of deleting the instantiating types from the transition labels of the system F transition graph. Or to put it another way: the system F transition labels are those of the untyped lambda transition graph together with type instantiations. The untyped lambda calculus games similar to those in [KNO02].

## 1.6 Related work

Affine linear polymorphism was modelled in [Abr97]<sup>14</sup> with a PER-like “intersections” of first-order games of the form [AJ94, AJM00]. Abramsky and Lenisa have explored systematic ways of modelling quantifiers so that, in the limited case in which all quantifiers are outermost (so in particular positive), models are fully complete [AL00]. (See subsection 1.4 for a simple example of a type at which full completeness fails.)

The hypergame/uniformity technique presented here has been applied to affine linear logic [MO01, Mur01], and has been used to study Curry-style type isomorphisms [dL06].

## 2 Transition system games and backtracking

A game such as Chess or **O's & X's** has a *state* (the configuration of the board or grid) and, for every state, a set of *transitions* or moves (*e.g.* ♖f3, ♜b4, **X** top-right, **O** centre), each with an ensuing state.<sup>15</sup> Such a game can be specified as a deterministic labelled *transition system*: an edge-labelled directed graph whose vertices are the states of the game, with a distinguished initial state. A fragment of the transition system for chess is illustrated below.<sup>16</sup>

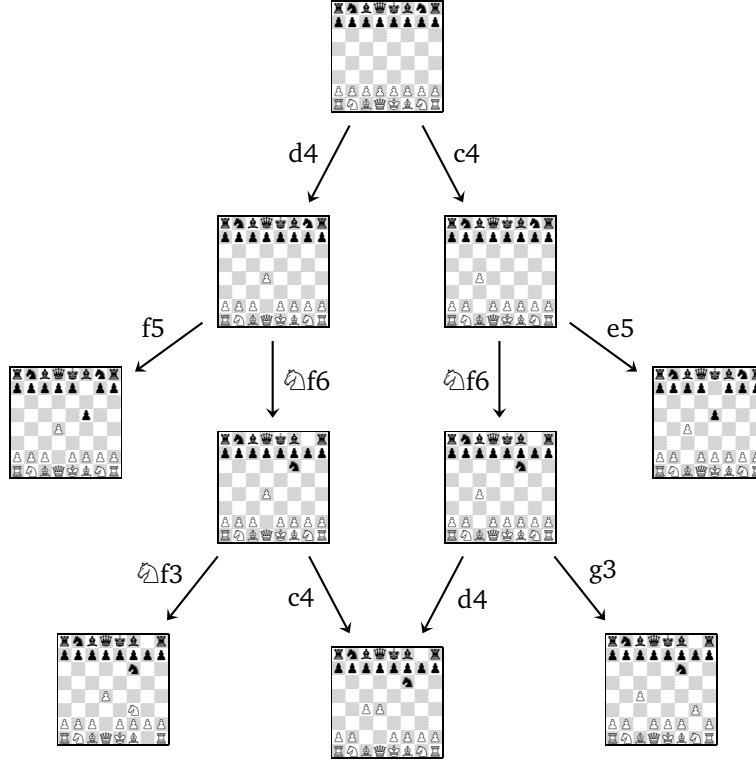
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<sup>13</sup>I have a vague recollection that just such an abstract machine was analysed for system F in the masters' thesis of Eike Ritter. I need to investigate this.

<sup>14</sup>Samson Abramsky's course at this summer school, during the summer before my D.Phil., is in part what inspired my choice of thesis topic.

<sup>15</sup>For a game of chance such as backgammon, one would specify a probability distribution over ensuing states, rather than a single ensuing state. We consider only deterministic games here.

<sup>16</sup>The states include data for en passant and castling and rights, and the turn (Black or White to move), not shown in the diagram.



Note that the graph is not a tree. Without a distinguished initial state, we shall refer to such a graph as a *transition graph*.

Formally, a **transition graph**  $(Q, M, \rightarrow)$  comprises a set  $Q$  of *states*, a set  $L$  of *labels*, and a partial **transition function**  $\rightarrow : Q \times L \rightarrow Q$ .<sup>17</sup> We write  $q \xrightarrow{l} q'$  for  $\rightarrow(q, l) = q'$ . A **transition system**  $(Q, L, \rightarrow, \star)$  is a transition graph  $(Q, L, \rightarrow)$  together with an *initial state*  $\star \in Q$ . A *trace* of  $(Q, L, \rightarrow, \star)$  is a finite sequence  $l_1 \dots l_k$  of labels  $l_i \in L$  ( $k \geq 0$ ) such that

$$\star \xrightarrow{l_1} q_1 \xrightarrow{l_2} q_2 \xrightarrow{l_3} \dots \xrightarrow{l_{k-1}} q_{k-1} \xrightarrow{l_k} q_k$$

for states  $q_i \in Q$  ( $1 \leq i \leq k$ ).<sup>18</sup> For example,  $d4 \text{ } \text{♞f6} \text{ } c4$  is a trace of chess, visible in the diagram above.

## 2.1 Games

Let  $M$  be a set of *moves*. A *trace over*  $M$  or *M-trace* is a list (finite sequence)  $m_1 \dots m_k$  of moves  $m_i \in M$  ( $k \geq 0$ ). A set  $G$  of  $M$ -traces is a *tree* if whenever  $m_1 \dots m_k$  is in  $G$  with  $k \geq 1$  then its *predecessor*  $m_1 \dots m_{k-1}$  is also in  $G$ , and the empty trace  $\varepsilon$  is in  $G$  (the root of the tree). A *game over*  $M$  or *M-game* is a tree of  $M$ -traces. Following [HO00], we write  $O$  for the first player (associated with odd moves, *i.e.*, moves in a trace with odd index), and  $P$  for the second player (associated with even moves).

<sup>17</sup>We write  $f : X \rightarrow Y$  if  $f$  is a partial function from  $X$  to  $Y$ , *i.e.*, a function  $X' \rightarrow Y$  for some  $X' \subseteq X$ .

<sup>18</sup>Note that the states  $q_i \in Q$  are uniquely determined by the  $l_i$ , since our transition systems are implicitly deterministic.

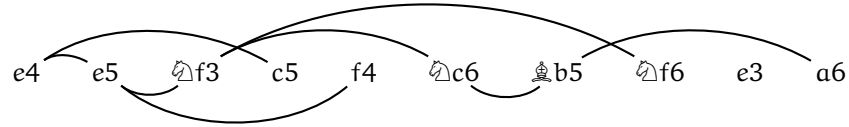
Every transition system  $\Delta$  with label set  $M$  defines a game  $G(\Delta)$  over  $M$ , namely the set of traces of  $\Delta$ . For example, if  $\Delta_{\text{ch}}$  is the chess transition system depicted above, and  $M_{\Delta}$  is the set of all chess moves  $\{\text{♞e5}, \text{♜a2}, \text{♝h1}, \dots\}$ , then  $G(\Delta_{\text{ch}})$  (the set of all traces of the chess transition system) is a game over  $M_{\Delta}$ . This game comprises all legal sequences of chess moves.

## 2.2 Strategies

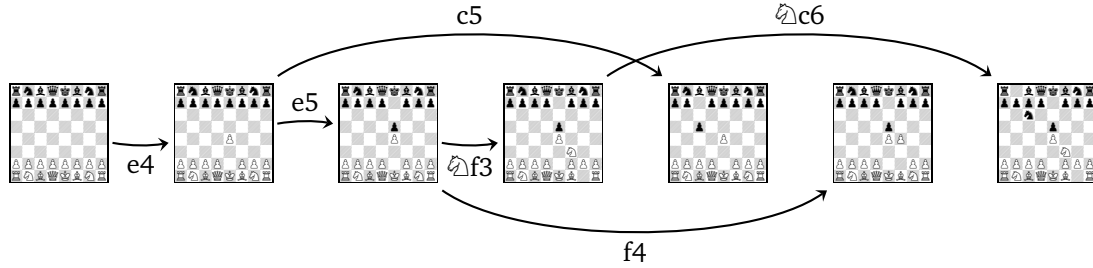
A **strategy** (implicitly for the second player  $P$ ) for a game  $G$  is a tree  $\sigma \subseteq G$  whose every odd-length trace has a unique one-move extension in  $\sigma$ : if  $m_1 \dots m_k \in \sigma$  and  $k$  is odd, there exists a unique move  $m$  such that  $m_1 \dots m_k m \in \sigma$ . A strategy  $\sigma$  for  $G$  is **live** (or **total**) if it responds to every stimulus: if  $m_1 \dots m_k \in \sigma$  with  $k$  even and  $m_1 \dots m_k m \in G$ , then  $m_1 \dots m_k m \in \sigma$ .<sup>19</sup>

## 2.3 Backtracking

When playing chess against a computer, there is usually an option to take a move back. If we allow both players (user and computer) to take back moves, and also to return to previously abandoned lines, we obtain a derived game in which a move is either an opening chess move (starting or restarting the game) or is a pair: a pointer to an earlier move by the opponent, and a chess move in response to that move. For example, here is a trace of backtracking chess, with time running left-to-right (so backtracking pointers are right-to-left):



The penultimate move  $e3$ , with no backtracking pointer, is a restarting move. Since this is a trace of a game with an underlying transition system, we can include the states in the depiction, as below, which corresponds to the first six moves above.



In this depiction we draw the pointers akin to transitions in the underlying transition system, with their labels. This clarifies the sense in which we refer back to a previous state during backtracking, and make our move from there.

We shall write  $\hat{G}$  for the backtracking variant of a game  $G$ , formalised below. Let  $M$  be a set of moves. A **dialogue** over  $M$  is an  $M$ -trace in which each element may carry an odd length pointer

<sup>19</sup>Thus  $m_1 \dots m_k m_n \in \sigma$  for a unique  $n$ , the “response of  $\sigma$  to  $m$  after  $m_1 \dots m_k$ ”. One is also tempted to call such a strategy *total*, by analogy with partial versus total functions; we shall stick with Conway’s original terminology [Con76].

to an earlier element (cf. [Lor60, Fel85, Coq91, Coq95, HO00]). For example, a dialogue over the set  $M_{\Delta}$  of chess moves is depicted above. Formally, a dialogue over  $M$  is an  $(\mathbb{N} \times M)$ -trace<sup>20</sup>

$$\langle \alpha_1, m_1 \rangle \dots \langle \alpha_k, m_k \rangle$$

such that  $i - \alpha_i \in \{1, 3, 5, \dots\}$  for  $1 \leq i \leq k$ . Each  $\alpha_i$  represents a pointer from  $m_i$  back to  $m_{\alpha_i}$ , with  $\alpha_i = 0$  coding “ $m_i$  has no pointer”. The formalisation of the chess dialogue depicted above is the following  $(\mathbb{N} \times M_{\Delta})$ -trace:

$$\langle 0, e4 \rangle \langle 1, e5 \rangle \langle 2, \textcircled{d}f3 \rangle \langle 1, c5 \rangle \langle 2, f4 \rangle \langle 3, \textcircled{d}c6 \rangle \langle 6, \textcircled{b}b5 \rangle \langle 3, \textcircled{d}f6 \rangle \langle 0, e3 \rangle \langle 7, a6 \rangle$$

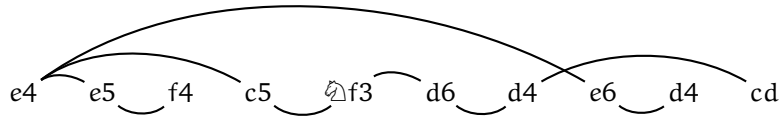
A move of the form  $\langle 0, m \rangle$ , without a pointer, is a **starting** move. A **thread** of a dialogue over  $M$  is any sequence of elements traversed from a starting move by following pointers towards the right. For example,  $e4 \ e5 \ \textcircled{d}f3 \ \textcircled{d}c6 \ \textcircled{b}b5$  is a thread of the chess dialogue above:



The singleton sequence  $e3$  is also a thread, as is  $e4 \ e5 \ f4$ . Formally, an  $M$ -trace  $m_{d_1} \dots m_{d_n}$  (where  $n \geq 0$ ) is a **thread** of the dialogue  $\langle \alpha_1, m_1 \rangle \dots \langle \alpha_k, m_k \rangle$  over  $M$  if  $\alpha_{d_1} = 0$  and  $\alpha_{d_j} = d_{j-1}$  for  $1 < j \leq n$ .

Let  $G$  be an  $M$ -game. A dialogue over  $M$  **respects**  $G$  if its threads are in  $G$ . For example, if Chess abbreviates our earlier formalisation  $G(\Delta_{\Delta})$  of the game of chess as a transition system game, then the dialogue over  $M_{\Delta}$  depicted above respects Chess (since every thread is a legal sequence of chess moves from the initial chess position). The **backtracking game**  $\widehat{G}$  is the set of all dialogues over  $M$  which respect  $G$ . For example, the dialogue over  $M_{\text{Chess}}$  depicted above is a trace of  $\widehat{\text{Chess}}$ , i.e., of “backtracking chess”.

The **P-backtracking game**  $\widehat{G}^P$  is obtained from  $\widehat{G}$  by permitting only the second player  $P$  to backtrack: every O-move (move in odd position) but the first points to the previous move. Formally,  $\widehat{G}^P$  comprises every  $\langle \alpha_1, m_1 \rangle \dots \langle \alpha_k, m_k \rangle$  in  $\widehat{G}$  such that  $\alpha_i = i - 1$  for all odd  $i \in \{1, \dots, k\}$ . A dialogue of  $\widehat{\text{Chess}}^P$  is shown below.



For every type  $T$  of system  $F$ , we shall define a transition system  $\Delta_T$  and define the **hypergame** associated with  $T$  simply as the backtracking game over this transition system, i.e.,  $\widehat{G}(\Delta_T)$ . For didactic purposes, we begin in the next section with the restricted case of lambda calculus.

### 3 Lambda calculus games

Let  $\lambda$  denote the types of  $\lambda$  calculus generated from a single base type  $X$  by implication  $\rightarrow$ . Every  $\lambda$  type  $T$  determines a transition system  $\Delta_T$ :

---

<sup>20</sup> $\mathbb{N} = \{0, 1, 2, \dots\}$ .

- States are  $\lambda$  types, with an additional initial state  $\star$ .
- A label is any  $i \in \{1, 2, 3, \dots\}$ , called a **branch choice**.
- Transitions. A 1-labelled transition

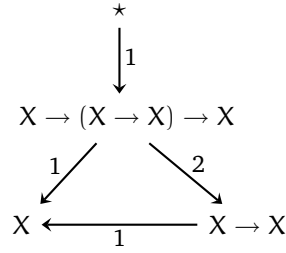
$$\star \xrightarrow{1} T$$

from the initial state to  $T$ , and transitions

$$T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_n \rightarrow X \xrightarrow{i} T_i$$

for  $1 \leq j \leq n$ .

For example, if  $U = X \rightarrow (X \rightarrow X) \rightarrow X$  then the reachable portion of the transition system  $\Delta_U$  is



so the associated (non-backtracking) game (set of traces)  $G(\Delta_U)$  is  $\{\varepsilon, 1, 11, 12, 121\}$ , where  $\varepsilon$  denotes the empty sequence.

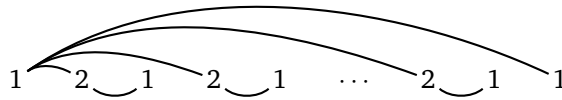
**THEOREM 1** *Let  $T$  be a lambda calculus type generated from a single base type  $X$  by implication  $\rightarrow$ . The  $\eta$ -expanded  $\beta$ -normal terms of type  $T$  are in bijection with finite live strategies on the  $P$ -backtracking game  $\widehat{G}(\Delta_T)^P$ .*

*Proof.* A routine induction: a restriction of the definability proof in [Hug97], in turn a variant of the definability proof in [HO00].  $\square$

The  $\eta$ -expanded  $\beta$ -normal forms  $t_n$  of  $U = X \rightarrow (X \rightarrow X) \rightarrow X$  (whose transition system was depicted above) are<sup>21</sup>

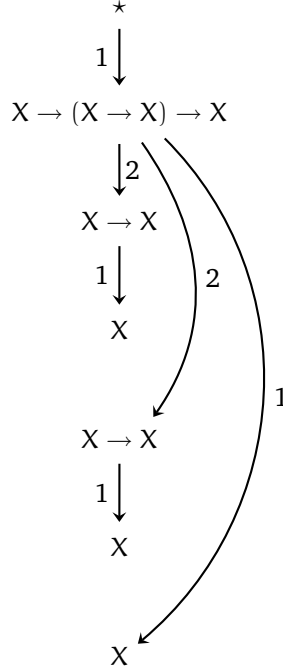
$$\lambda x^X. \lambda f^{X \rightarrow X}. f^n(x)$$

for  $n \geq 0$  and the unique maximal trace of the corresponding live finite strategy  $\tau_n$  on  $\widehat{G}(\Delta_T)^P$  is



with  $n$  occurrences of 2. Below we depict this dialogue in the case  $n = 2$  (corresponding to the term  $\lambda x^X. \lambda f^{X \rightarrow X}. f(f(x))$ ) with its states (as we did for the chess dialogue on page 21). It is easier to display with time running down the page, rather than from right to left.

<sup>21</sup>  $f^n$  denotes  $n$  applications of  $f$ :  $f^0(x) = x$  and  $f^n(x) = f(f^{n-1}(x))$  for  $n \geq 1$ .



This notation highlights the similarity with Lorenzen's dialogues [Lor60, Fel85]. What we show as states, he referred to as *assertions*.

### 3.1 The copycat condition

In this section we introduce the **copycat condition** [Hug97] on strategies, which is crucial for uniformity (more precisely, for us to be able to implement uniformity via *copycat expansion* later). This condition is a slight restriction of a condition of Lorenzen for dialogue games (listed as condition (D10) in [Fel85]). We shall introduce the condition in the context of lambda calculus games; the generalisation to system F games in the sequel is immediate.

Extend the set  $\lambda$  of lambda calculus types from the previous subsection to those generated by implication  $\rightarrow$  from the ambient set  $\text{Var}$  of system F type variables. The transition system  $\Delta_T$  associated with a  $\lambda$  type  $T$  is defined exactly as in the previous section, but now in the transitions

$$T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_n \rightarrow X \quad \xrightarrow{i} \quad T_i$$

$X$  may be *any* type variable in  $\text{Var}$ .

The **colour** of a transition

$$T_1 \rightarrow \dots \rightarrow T_n \rightarrow X \quad \xrightarrow{i} \quad U_1 \rightarrow \dots \rightarrow U_m \rightarrow Y$$

(where necessarily  $T_i = U_1 \rightarrow \dots \rightarrow U_m \rightarrow Y$ ) is the rightmost variable  $Y$  in the target. The colour of a move in a trace of  $G(\Delta_T)$  or a dialogue in  $\widehat{G(\Delta_T)}$  is the colour of the associated transition. A dialogue in the P-backtracking game  $\widehat{G(\Delta_T)}^P$  satisfies the **copycat condition** if the colour of every

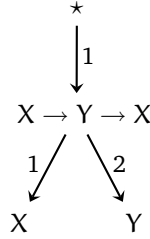


P-move (even-index move) is equal to the colour of the preceding O-move.<sup>22</sup> A strategy satisfies the copycat condition if each of its traces satisfies the copycat condition.

As a simple illustration of the copycat condition, consider the type

$$\mathbf{U} = X \rightarrow Y \rightarrow X$$

whose transition system  $\Delta_{\mathbf{U}}$  is below (only reachable states shown).



The colour of the top and lower-left transitions is X, and the colour of the lower-right transition is Y. The associated (non-backtracking) game (set of traces)  $G(\Delta_{\mathbf{U}})$  is  $\{\epsilon, 1, 2, 11, 12\}$ . There are two live strategies in the P-backtracking game  $\widehat{G(\Delta_{\mathbf{U}})}^P$ , whose maximal traces are as follows, with the colour of each move shown beneath it in brackets:

$$\begin{array}{cc} 1 & 1 \\ \text{---} & \text{---} \\ (X) & (X) \end{array} \qquad \begin{array}{cc} 1 & 2 \\ \text{---} & \text{---} \\ (X) & (Y) \end{array}$$

The first strategy satisfies the copycat condition, while the second does not. The strategies correspond (respectively) to the terms

$$\lambda x^X. \lambda y^Y. x \qquad \lambda x^X. \lambda y^Y. y$$

of which only the former is typed correctly as  $X \rightarrow Y \rightarrow X$ . The second attempts to return  $y$  of type  $Y$ , while the rightmost variable of  $X \rightarrow Y \rightarrow X$  is  $X$ . This corresponds to the failure of the copycat condition for the second strategy.

The following is a corollary of the theorem above.

**THEOREM 2** *Let  $T$  be a lambda calculus type generated from the set  $\text{Var}$  of system  $F$  type variables by implication  $\rightarrow$ . The  $\eta$ -expanded  $\beta$ -normal terms of type  $T$  are in bijection with finite live strategies on the P-backtracking game  $\widehat{G(\Delta_T)}^P$  which satisfy the copycat condition.*

### 3.2 Remarks on Hyland-Ong arenas

*This section is for readers familiar with Hyland-Ong games [HO00]. It can be skipped without loss of continuity.*

The set of non-empty traces of a transition system forms a forest under the prefix order, and is therefore an **arena** in the sense of Hyland and Ong [HO00]. Write  $A(\Delta)$  for the arena of a

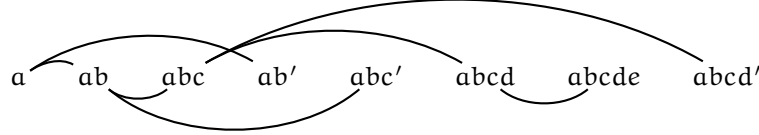
<sup>22</sup>Lorenzen's condition (D10) required the colour to be equal to *any* prior O-move in a P-backtracking trace.

transition system  $\Delta$ , and for a lambda calculus type  $T$  abbreviate  $A(\Delta(T))$  to  $A(T)$ . The following arena isomorphism is immediate:

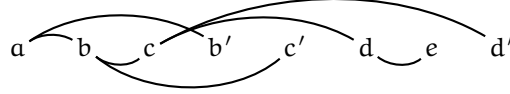
$$A(T \rightarrow U) \cong A(T) \Rightarrow A(U)$$

where  $\Rightarrow$  is the Hyland-Ong function space operation on arenas and  $\cong$  is isomorphism of forests.

Elements of these arenas are sequences (traces), and therefore Hyland-Ong dialogues in them suffer some redundancy, as in (for example)



in the arena generated by a transition system with transition labels  $a, b, b', c, c', d, d'$ , whose traces include  $abcde, abcd', abc', etc.$  Clearly, one can abbreviate this trace to



eliminating the redundancy. This is how we have opted to formalise the backtracking games over transition systems in the previous subsections. Note, however, this notational difference is trivial, and in spirit they are essentially Hyland-Ong arena/dialogue games. The notation is simply tailored towards arenas whose forests are described as sets of traces, rather than partial-order forests as used originally by Hyland and Ong [HO00]. Since our games are Hyland-Ong games, and we have the isomorphism relating syntactic  $\rightarrow$  with arena  $\Rightarrow$  above, we obtain composition (hence a category) as standard Hyland-Ong composition of innocent strategies.

In the next section we extend the lambda calculus transition systems to system F transition systems. The following arena (forest) isomorphisms will then hold:

$$\begin{aligned} A(T \rightarrow U) &\cong A(T) \Rightarrow A(U) \\ A(\forall X.T) &\cong \prod_{\text{Types } U} A(T[U/X]) \end{aligned}$$

The arena-product  $\prod$  (disjoint union of forests) is taken over all system F types. Composition in our system F model is simply Hyland-Ong composition.

## 4 System F games (hypergames)

We extend the lambda calculus transition systems defined above to all of system F. States will be types, as before, and a transition will remain a branch choice  $i \geq 1$ , but now together with some types to instantiate quantifiers. We begin by precisely defining quantifier instantiation.

Recall that a **prenex type** is a type in which all quantifiers have been pulled to the front by exhaustively applying the rewrite

$$T \rightarrow \forall X.U \rightsquigarrow \forall X.T \rightarrow U$$

throughout the type.<sup>23</sup> Thus a type is prenex if and only if it has the form

$$\forall X_1. \forall X_2. \dots \forall X_m. T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_n \rightarrow X$$

for prenex types  $T_i$  and type variables  $X$  and  $X_j$ . Write  $T[V/X]$  for the result of substituting the type  $V$  for the free variable  $X$  throughout the prenex type  $T$ , and (if necessary) converting to prenex form. For example

$$(X \rightarrow X)[\forall Y. Y/X] = \forall Y. (\forall Y'. Y') \rightarrow Y$$

via:

$$(X \rightarrow X)[\forall Y. Y/X] \xrightarrow{\text{substitute}} (\forall Y. Y) \rightarrow (\forall Y. Y) \xrightarrow{\text{prenex}} \forall Y. (\forall Y'. Y') \rightarrow Y.$$

Define

$$\forall X. T \cdot V = T[V/X]$$

called the result of **importing**  $V$  into  $\forall X. T$ . For example,

$$\forall X. X \rightarrow X \cdot \forall Y. Y = \forall Y. (\forall Y'. Y') \rightarrow Y.$$

Write  $T \cdot V_1 V_2 \dots V_n$  for the iterated importation  $(\dots ((T \cdot V_1) \cdot V_2) \cdot \dots) \cdot V_n$ , when defined. For example,

$$\forall X. X \rightarrow X \cdot (\forall Y. Y) (Z \rightarrow Z) = (\forall Y. Y) \rightarrow Z \rightarrow Z.$$

A prenex type is **resolved** if it has no outermost quantifier, *i.e.*, it has the form

$$T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_n \rightarrow X,$$

a form which we shall often abbreviate to

$$T_1 T_2 \dots T_n \rightarrow X$$

Each  $T_i$  is called a **branch**. If  $T \cdot U_1 \dots U_n$  is resolved, we say that  $U_1 \dots U_n$  **resolves**  $T$  to  $T \cdot U_1 \dots U_n$ . For example, we saw above that  $(\forall Y. Y)(Z \rightarrow Z)$  resolves  $\forall X. X \rightarrow X$  to  $(\forall Y. Y) \rightarrow Z \rightarrow Z$ .

Define the transition system  $\Delta_T$  of a prenex type  $T$  as follows:

- States are resolved prenex types, with an additional initial state  $\star$ .<sup>24</sup>
- A label is a pair  $\langle i, V_1 \dots V_k \rangle$  where  $i \geq 1$  is a **branch choice**,  $k \geq 0$  and each  $V_i$  is a type, called an **import**.
- Transitions. A 1-labelled transition

$$\star \xrightarrow{1} T$$

from the initial state to  $T$ , and transitions

$$T_1 T_2 \dots T_n \rightarrow X \xrightarrow{\langle i, V_1 \dots V_k \rangle} U_1 U_2 \dots U_m \rightarrow Y$$

whenever  $1 \leq i \leq n$  and

$$T_i \cdot V_1 \dots V_k = U_1 U_2 \dots U_m \rightarrow Y$$

(Thus a transition chooses a branch  $T_i$  and resolves it to form the next state.)

<sup>23</sup>Without loss of generality, in the rewrite assume  $X$  is not free in  $T$ .

<sup>24</sup>Prenex types were drawn graphically in [Hug97, Hug00], in a manner akin to Böhm trees, and called *polymorphic arenas*.

More generally, the transition system of a type is the transition system of its prenex form.

An example transition is shown below.

$$(X' \rightarrow X' \rightarrow X') \rightarrow (\forall X.X \rightarrow X) \rightarrow X'' \quad \langle 2, (\forall Y.Y)(Z_1 \rightarrow Z_2 \rightarrow Z) \rangle \quad (\forall Y.Y) \rightarrow Z_1 \rightarrow Z_2 \rightarrow Z$$

The branch choice 2 selects the branch  $\forall X.X \rightarrow X$  and the imports  $\forall Y.Y$  and  $Z_1 \rightarrow Z_2 \rightarrow Z$  resolve this branch to form the next state.<sup>25</sup>

#### 4.1 Implicit prenexification

Prenexification is a lynchpin of the hypergames approach [Hug97]: it is critical to the dynamics of hypergames that in a type  $T \rightarrow \forall X.U$  the quantifier  $\forall X$  is available for instantiation. Whether we make the prenexifications  $T \rightarrow \forall X.U \rightsquigarrow \forall X.T \rightarrow U$  explicit during play or not is optional. We can just as well leave prenexification implicit, by formally designating  $\forall X$  as available for instantiation in  $T \rightarrow \forall X.U$ .

A quantifier  $\forall X$  in a type  $T$  is **available** if  $T$  has any of the following forms:<sup>26</sup>

- $T = \forall X.U$
- $T = U \rightarrow T'$  and  $\forall X$  is available in  $T'$
- $T = \forall Y.T'$  and  $\forall X$  is available in  $T'$ .

For example,  $\forall X$  and  $\forall Y$  are available in  $\forall Y.(\forall Z.Y \rightarrow Z) \rightarrow \forall X.X$ , but  $\forall Z$  is not.<sup>27</sup>

Type resolution and importation are tweaked in the obvious way, as follows. A type is **resolved** if it has no available quantifier, *i.e.*, if it has the form

$$T_1 \rightarrow T_2 \rightarrow \dots \rightarrow T_n \rightarrow X \quad = \quad T_1 \dots T_n \rightarrow X$$

for  $n \geq 0$  and types  $T_i$ , called **branches**. (All we have done is drop the requirement that the  $T_i$  be prenex.) Let  $\forall X$  be the leftmost available quantifier in a type  $T$ , and let  $T^X$  be the result of deleting  $\forall X$  from  $T$  (*e.g.* if  $T = U \rightarrow \forall X.V$  then  $T^X = U \rightarrow V$ ). Define

$$T \cdot V \quad = \quad T^X[V/X],$$

the result of **importing** a type  $V$  into  $T$ , and define **iterated importation**  $T \cdot V_1 \dots V_n$  as before.

The (lazy style) transition system  $\Delta_T$  of a system  $F$  type remains essentially unchanged:

- States are system  $F$  types, with an additional initial state  $\star$ .
- A label is a pair  $\langle i, V_1 \dots V_k \rangle$  where  $i \geq 1$  is a **branch choice**,  $k \geq 0$  and each  $V_i$  is a type, called an **import**.

<sup>25</sup>To obtain a category with products, we extend system  $F$  with products, and allow import/resolution/substitution with products.

<sup>26</sup>We assume without loss of generality throughout this section that all bound variables are distinct from one another and from the free variables.

<sup>27</sup>Note that  $\forall X$  is available in  $T$  iff it is one of the outermost quantifiers in the prenex form  $\tilde{T}$  of  $T$  (*i.e.*,  $\tilde{T} = \forall X_1 \dots \forall X_k.U$  and  $X$  is among the  $X_i$ ). In this sense, prenexification is implicit, or “lazy”: from a behavioural point of view, we’re still working with prenex types.

- Transitions. A 1-labelled transition

$$\star \xrightarrow{1} T$$

from the initial state to  $T$ , and transitions

$$T_1 T_2 \dots T_n \rightarrow X \xrightarrow{\langle i, V_1 \dots V_k \rangle} U_1 U_2 \dots U_m \rightarrow Y$$

whenever  $1 \leq i \leq n$  and

$$T_i \cdot V_1 \dots V_k = U_1 U_2 \dots U_m \rightarrow Y$$

## 5 Black box characterisation of system F terms

A **black box importation** is an importation of the form

$$\forall X. T \cdot X = T,$$

simply deleting the quantifier. Thus the bound variable  $X$  becomes free. We refer to  $X$  as a **black box**. (We continue to assume, without loss of generality, that within a type all bound variables are distinct from one another and from all free variables.) Let  $T$  be a closed<sup>28</sup> system F type and  $d$  a dialogue in the P-backtracking game  $\widehat{G(\Delta_T)}^P$ . The first player  $O$  **imports black boxes** in  $d$  if every importation associated with  $O$  in  $d$  is a black box importation, and the second player  $P$  **respects black boxes** in  $d$  if every import associated with  $P$  takes its free variables among the black boxes imported hitherto by  $O$ . A dialogue in which  $O$  imports black boxes and  $P$  respects them is a **black box dialogue**. The **black box game**  $\widehat{G(\Delta_T)}^B$  is the restriction of the P-backtracking game  $\widehat{G(\Delta_T)}^P$  to black box dialogues.

The copycat condition extends from the lambda calculus case to system F in the obvious way: the colour of a transition is once again the rightmost variable of the target.

**THEOREM 3** *The  $\eta$ -expanded  $\beta$ -normal terms of a closed system F type  $T$  are in bijection with finite live strategies on the black box game  $\widehat{G(\Delta_T)}^B$  which satisfy the copycat condition.*

*Proof.* The definability proof in [Hug97]. □

## 6 Uniformity by copycat expansion

The black box game is highly unsymmetric:

- (1)  $P$  can backtrack, while  $O$  cannot.
- (2)  $P$  is subject to the copycat condition, while  $O$  is not.
- (3)  $O$  can only import black boxes (free variables);  $P$  can import arbitrary types, so long as their free variables are prior black boxes.

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<sup>28</sup>No free variables.

To compose strategies we must symmetrise the game, so that O and P can interact.

A symmetrisation of backtracking (1) was obtained by Coquand [Coq91, Coq95]. A shared history of two asymmetric strategies is built, in which both players backtrack. Each time either asymmetric strategy plays a move, it can only see a projection of the shared history in which the opposing player does not backtrack. This interaction was made lambda-calculus specific by Hyland-Ong [HO00], who called the projections *views* and called the symmetrised strategies *innocent*.

Symmetrising the copycat condition (2) will be automatic, coming as a simple side effect of the views: we simply demand that, in their respective views, both strategies adhere to the copycat condition.

We shall symmetrise with respect to black boxes (3) via the notion of *copycat expansion* [Hug06] recalled below.<sup>29</sup>

Symmetrising (1) yields interaction for lambda calculus over a single base type symbol. Symmetrising (1) & (2) yields interaction for lambda calculus over a set of base type symbols. Symmetrising (1)—(3) yields interaction for system F.

Symmetrising	yields interaction for
(1)	$\lambda$ , single base type
(1) & (2)	$\lambda$ , set of base types
(1) & (2) & (3)	system F

We shall refer to the copycat condition together with copycat expansion as **uniformity**. A visual summary of the symmetrisation is below, where T is a system F type.

$$\widehat{G(\Delta_T)}^B \xrightarrow{\text{uniformity}} \widehat{G(\Delta_T)}^P \xrightarrow{\text{innocence}} \widehat{G(\Delta_T)}$$

An arrow here indicates how a strategy on the left lifts to a strategy on the right.

### 6.1 Symmetrising black boxes via copycat expansion

Let T be a closed system F type and d a dialogue in the P-backtracking game  $\widehat{G(\Delta_T)}^P$  which satisfies the copycat condition. Let X be a black box in d, let U be a type, and define  $d[U/X]$  as the result of substituting U for free occurrences of X in the imports of d.

To be continued...

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<sup>29</sup>Copycat expansion was implicit in [Hug00], occurring during interaction. In [Hug06] it was made explicit, being applied to the strategies prior to interaction, rather just during interaction.

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